Jour. Ind. Soc. Ag. Statistics
***44** (1), 1992 : 1–14*

Preliminary Test Estimators of Regression Coefficient

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(Received : September, 1981/87)

Summary

Two preliminary test estimators of regression coefficient have been suggested in the linear model $Y = \gamma + \beta x + \varepsilon$, assuming (X,Y) to follow a bivariate normal distribution. Preliminary test of equality of variances of X and Y is conducted using Morgan's t-test[16] and suitable estimator is chosen depending on the outcome of the preliminary test. Bias and MSE of the proposed estimators are derived. Recommendations on the choice of sample size and level of preliminary test are made on the basis of numerical values of bias and relative efficiency with respect to sample regression coefficient.

Key words : Linear model, Morgan's t-test, Wishart density, Hypergeometric function, Beta function, Kamp de Feriet function, Preliminary test.

Introduction

After the leading paper by Bancroft [4], various preliminary test (PT) procedures have been studied by Mosteller [17], Han and Bancroft [7], Al-Bayyati and Arnold [3], Ahsanullah [1] and others. These procedures have been applied in regression analysis also.

One group of researchers have used PT to decide about the number of regressor variables to be retained in the model (cf. Kitagawa [11], Larson and Bancroft [13] [14], Kennedy and Bancroft [10] etc. Another group of researchers used PT to judge the validity of some prior knowledge about the model (Rahim [18]. Johnson, et. al [9] have used PT of equality of two regression lines to decide whether to pool or not to pool the second data also for estimating the regression line for one data set. Relative efficiency of the proposed estimator is obtained and recommendations on the choice of levels of PT are made so that at least a specified efficiency is attained.

Estimation of regression coefficient and intercepts after a PT of parallelism of several regression lines has been considered by Han and Bancroft [8] and Lambert et. al. [12]. Akritas et. al. [2] have considered a non parametric approach to the same problem.

Shukla [26] has considered the case when two random samples of different sizes are available from two linear models, with X, a known or controlled variable and the dependent variable Y having equal variance σ^2.

In this paper the simple linear regression model $y = \gamma + \beta x + \varepsilon$ has been considered; assuming that (x, y) has bivariate normal distribution with parameters μ_x, μ_y, σ_x^2, σ_y^2 and ρ. The estimate of y (the variable under study), depends on the estimated value of γ and β. Some times y may be such that its variability is of the same order as that of independent variable x. For example, it is quite possible that variances of length and breadth of leaves (say of plum or beatle), the breadth and circumference of human head, are equal.

For such variables $\sigma_x = \sigma_y$ so $\beta = \rho$. Hence the sample correlation coefficient (r) seems to be an appropriate estimator of β. However, though r seems to be appropriate, the m.l.e. of ρ, under this condition turns out to be $2s_{12}/(s_1^2 + s_2^2)$ (cf. Mehta and Gurland [15].

If $\sigma_x \neq \sigma_y$ the m.l.e. for β is the well known sample regression coefficient b. In practice as one is not sure of the equality of σ_x and σ_y, an uncertainty prevails regarding the choice of an estimator for β. So first a preliminary test of $H_0 : \sigma_x = \sigma_y$ is conducted on the basis of available sample and the estimator is chosen according to the outcome of the test. Two PT estimators $\hat{\beta}$ and β have been proposed, using r and $2s_{12}/(s_1^2 + s_2^2)$ respectively and integral expressions for the bias and mean squared error (MSE) have been obtained. Series expressions have also been obtained. In addition, numerical comparison of the two estimators with the usual estimator b has been undertaken.

2. *Proposed Preliminary Test Estimators*

Consider a random sample (x_i, y_i), i = 1, 2, . . . N from a bivariate normal distribution with parameters μ_x, μ_y, σ_x^2, σ_y^2 and ρ. Let $y = \gamma + \beta x + \varepsilon$ be the regression model and it is desired to estimate β. In practice one may suspect the equality of the variances from the nature of the data. So first a test of $H_0 : \sigma_x^2 = \sigma_y^2$ may be performed to resolve the uncertainty, using the t-test suggested by Morgan [16]. This test rejects H_O at level α if

$$\left| \frac{(s_1^2 - s_2^2)\sqrt{N-2}}{2\left(s_1^2 s_2^2 - s_{12}^{1/2}\right)^{1/2}} \right| > t_{\alpha/2}$$

where $s_1^2 = \frac{1}{N}\sum_{i=1}^{N} (x_i - \bar{x})^2$, $\quad s_2^2 = \frac{1}{N}\sum_{i=1}^{N} (y_i - \bar{y})^2$,

$$s_{12} = \frac{1}{N}\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y}) , \quad \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i , \quad \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$$

and $t\alpha/2$ is the upper $100\alpha/2$ percent point of students' t-distribution with N-2 d.f.

If H_O is rejected the usual sample regression coefficient b is used. If H_O is accepted the sample correlation coefficient may be used because it is an estimator of ρ. So the first estimator of β is proposed as

$$\hat{\beta} = \begin{cases} r & \text{if } H_O : \sigma_x = \sigma_y \text{ is accepted} \\ b & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{s_{12}}{s_1 s_2} & \text{if } \left| \dfrac{(s_1^2 - s_2^2)\, N-2}{2\,(s_1^2 s_2^2 - s_{12}^2)^{1/2}} \right| < t_{\alpha/2} \\ \dfrac{s_{12}}{s_1^2} & \text{otherwise} \end{cases} \qquad (2.1)$$

As m.l.e.'s have many optimum properties, one may use $\frac{2s_{12}}{(s_1^2 + s_2^2)}$ instead of r, if H_O is accepted with the hope of larger gain in efficiency. So a second estimator of β may be proposed as

$$\beta = \begin{cases} \dfrac{2s_{12}}{s_1^2 s_2^2} & \text{if } \left| \dfrac{(s_1^2 - s_2^2)\, N-2}{2\,(s_1^2 s_2^2 - s_{12}^2)^{1/2}} \right| < t_{\alpha/2} \\ \dfrac{s_{12}}{s_1^2} & \text{otherwise} \end{cases} \qquad (2.2)$$

3. *Bias and MSE of Estimators*

For evaluating the bias and MSE of $\hat{\beta}$ and β one may use the following result :

Lemma. Let T, T_1 and T_2 are functions of random vector X with joint p.d.f. $p(\underline{x})$ such that

$$T = \begin{cases} T_1 & \text{if } \underline{X} \in A \\ T_2 & \text{otherwise} \end{cases}$$

Then

$$E(T) = E(T_2) + \int_A \cdots \int (T_1 - T_2)\ p(\underline{x})\, d\underline{x} \qquad (3.1)$$

Proof : It can be easily seen that

$$E(T) = \int_A \cdots \int (T_1)\ p(\underline{x})\, d\underline{x} + \int_{A^c} \cdots \int (T_2)\ p(\underline{x})\, d\underline{x} = \text{r.h.s. of (3.1).}$$

Both $\hat{\beta}$ and β are of the same form as T with A replaced by R, the acceptance region of H_O. The joint density of s_1^2 , s_2^2 , s_{12} is given by the Wishart density (cf. Mehta and Gurland (1969)

$$f(s_1^2 , s_2^2 , s_{12}) = K (s_1^2\ s_2^2 - s_{12}^2)^{\frac{1}{2}(N-4)}$$

$$\exp\left[-\frac{1}{2} N(1 - \rho^2)^{-1} \frac{s_1^2}{\sigma_x^2} + \frac{s_2^2}{\sigma_y^2} - 2\rho \frac{s_{12}}{\sigma_x \sigma_y} \right].$$

where

$$K = \frac{N^{N-1}}{4\pi \Gamma(N-2) \left\{\sigma_x^2\ \sigma_y^2\ (1 - \rho^2)\right\}^{\frac{1}{2}(N-1)}} \qquad (3.2)$$

Hence using (3.1) one can write

$$E(\hat{\beta}) = \left(\frac{s_{12}}{s_1^2}\right) + \iiint_R \left(\frac{s_{12}}{s_1 s_2}\right)\left(1 - \frac{s_2}{s_1}\right) f(s_1^2 , s_2^2\ s_{12})\, ds_1^2\, ds_2^2\, ds_1$$

$$= E(b) + I_1' = \beta + I_1' \qquad (3.3)$$

$$E(\hat{\beta}^2) = E(b^2) + \iiint_R \frac{s_{12}^2}{s_1^2} \left(\frac{1}{s_2^2} - \frac{1}{s_1^2}\right) f(s_1^2, s_2^2, s_{12})\, ds_1^2\, ds_2^2\, ds_{12}$$

$$= E(b^2) + I_2' = \frac{\sigma_y^2 (1-\rho^2)}{\sigma_x^2 (N-3)} + \beta^2 + I_2' \tag{3.4}$$

$$E(\hat{\beta}) = E(b) + \iiint_R s_{12} \left(\frac{2}{s_1^2 + s_2^2} - \frac{1}{s_1^2} \right) f(s_1^2, s_2^2, s_{12})\, ds_1^2\, ds_2^2\, ds_{12}$$

$$= E(b) + J_1' = \beta + J_1' \tag{3.5}$$

$$E(\hat{\beta}^2) = E(b^2) + \iiint_R s_1^2 \left(\frac{4}{(s_1^2 + s_2^2)^2} - \frac{1}{s_1^4} \right) f(s_1^2, s_2^2, s_{12})\, ds_1^2\, ds_2^2\, ds_{12}$$

$$= \frac{\sigma_y^2 (1-\rho^2)}{\sigma_x^2 (N-3)} + \beta^2 + J_2' \tag{3.6}$$

Applying the transformations TR_1 and TR_2 given by

$$TR_1 : u = \frac{2s_{12}}{s_1^2 + s_2^2}, \quad v = s_1^2 + s_2^2 \text{ and } w = s_2^2 \tag{3.7}$$

$$TR_2 : u = u, \quad v = v \text{ and } w = \frac{1}{2} v (1 + t\sqrt{1-u^2}) \tag{3.8}$$

successively on I_1' in (3.3) we get

$$I_1' = \frac{K}{2^{N-2}} \int_{-h}^{h} \int_{-1}^{1} \int_{0}^{\infty} \varphi_3(u, t)\, v^{N-2} \exp[-a\, \varphi_4(u, t)\, v]\, dv\, du\, dt$$

where

$$\varphi_3(u, t) = u (1-u^2)^{\frac{1}{2}(N-3)} \left[1 - \left\{ \frac{1 + t\sqrt{1-u^2}}{1 - t\sqrt{1-u^2}} \right\}^{\frac{1}{2}} \right]$$

$$(1 - t^2)^{½(N-4)} [1 - t^2 (1-u^2)]^{-½},$$

$$\varphi_4(u, t) = d + c\, t\sqrt{1-u^2} + gu, \; a = N/4 (1-\rho^2)\, \sigma_x^2, \; d = \left(1 + \frac{\sigma_x^2}{\sigma_y^2} \right),$$

$$c = \left(\frac{\sigma_x^2}{\sigma_y^2} - 1\right), \; g = \frac{-2\,\rho\sigma_x}{\sigma_y} \text{ and } h^2 = \frac{t_{\alpha/2}^2}{t_{\alpha/2}^2 + N-2} \tag{3.9}$$

Integrating innermost gamma integral w.r.t.v and substituting t=hz, (3.9) can be reduced to even more convenient form for Gauss-quadrature formula as

$$I_1' = \frac{K\Gamma\,(N-1)\,h}{2^{N-2}\,a^{N-1}} \int_{-1}^{1}\int_{-1}^{1} \frac{\varphi_3\,(u,\,hz)}{[\varphi_4\,(u,\,hz)]^{N-1}}\,du\,dz = C_k\,I_1\,, \tag{3.10}$$

$$C_k = (N-2)\,2^{N-2} \left\{\frac{\sigma_x^2\,(1-\rho^2)}{\sigma_y^2}\right\}^{\frac{1}{2}N-1} \frac{h}{\pi} \tag{3.11}$$

Thus integral expression for bias of $\hat{\beta}$ is obtained. Following same lines after applying TR_1 and TR_2 on I_2', J_1' and J_2' integrating over v as gamma integral and substituting t=hz

$$I_2' = C_k \int_{-1}^{1}\int_{-1}^{1} \frac{\varphi_6\,(u,\,hz)}{[\varphi_4\,(u,\,hz)]^{N-1}}\,du\,dz = C_k\,I_2\,, \tag{3.12}$$

$$J_1' = -h\,C_k \int_{-1}^{1}\int_{-1}^{1} \frac{\varphi_7\,(u,\,hz)}{[\varphi_4\,(u,\,hz)]^{N-1}}\,du\,dz = -h\;C_k\,J_1 \tag{3.13}$$

$$J_2' = -h\,C_k \int_{-1}^{1}\int_{-1}^{1} \frac{\varphi_8\,(u,\,hz)}{[\varphi_4\,(u,\,hz)]^{N-1}}\,du\,dz = -h\;C_k\,J_2 \tag{3.14}$$

where

$$\varphi_6(u,\,t) = u^2(1-u^2)^{\frac{1}{2}(N-3)}\,(1-t^2)^{\frac{1}{2}(N-4)}\left(\frac{1-t\sqrt{1-u^2}}{1+t\sqrt{1-u^2}} - 1\right)(1-t\sqrt{1-u^2})^{-2}$$

$$\varphi_7(u,\,t) = u\,(1-u^2)^{\frac{1}{2}(N-3)}\;(-t)\,(1-t^2)^{\frac{1}{2}(N-4)}\,(1-t\sqrt{1-u^2})^{-1}$$

$$\varphi_8(u,\,t) = u^2(1-u^2)^{\frac{1}{2}(N-3)}\,(-t)\,(1-t^2)^{\frac{1}{2}(N-4)}(2-t\sqrt{1-u^2})(1-t\sqrt{1-u^2})^{-2}$$

(3.12) gives the integral expression for bias of β.

Now $MSE(\hat{\beta}) = E(\hat{\beta} - \beta)^2 = E(\hat{\beta}^2) + \beta^2 - 2\beta E(\hat{\beta})$.

Using (3.3), (3.4), (3.10) and (3.12) integral expression for

$$MSE(\hat{\beta}) = \frac{\sigma_y^2 (1-\rho^2)}{\sigma_x^2 \ (N-3)} + C_k I_2 - 2\rho \left(\frac{\sigma_y}{\sigma_x}\right) C_k I_1 \qquad (3.15)$$

and from (3.5), (3.6), (3.13) and (3.14) integral expression for

$$MSE(\beta) = \frac{\sigma_y^2 (1-\rho^2)}{\sigma_x^2 \ (N-3)} - h C_k J_2 + 2\rho \left(\frac{\sigma_y}{\sigma_x}\right) h C_k J_1 \qquad (3.16)$$

Series expressions can also be obtained by expanding $\exp[-a\varphi_4(u,t)v]$ before integrating on v. The expressions are quite lengthy so are given in Appendix alongwith their derivation.

4. Relative Efficiencies of $\hat{\beta}$ and β

Let e_1 and e_2 denote the relative efficiencies of $\hat{\beta}$ and β with respect to b. From (3.15)

$$e_1 = \frac{V(b)}{MSE(\hat{\beta})} = \frac{\dfrac{\sigma_y^2 (1-\rho^2)}{\sigma_x^2 (N-3)}}{\dfrac{\sigma_y^2 (1-\rho^2)}{\sigma_x^2 (N-3)} + C_k I_2 - 2\rho (\sigma_y/\sigma_x) C_k I_1} \qquad (3.17)$$

and from (3.16)

$$e_2 = \frac{V(b)}{MSE(\beta)} = \frac{\dfrac{\sigma_y^2 (1-\rho^2)}{\sigma_x^2 (N-3)}}{\dfrac{\sigma_y^2 (1-\rho^2)}{\sigma_x^2 (N-3)} - h \ C_k J_2 + 2\rho (\sigma_y/\sigma_x) h C_k J_1}$$

Thus integral expressions for relative efficiencies of $\hat{\beta}$ and β with respect to b are obtained.

5. Numerical Findings

To calculate bias and efficiencies of the estimatiors $\hat{\beta}$ and β the integrals I_1, I_2, J_1 and J_2 have been evaluated by applying 20 points

Gauss-Legendre quadrature formulae of the inner and the outer integrals.

The numerical values of the bias and efficiencies of the two estimators have been obtained for N= 5, 10, 20, 30; $\frac{\sigma_x^2}{\sigma_y^2}$ = .25(.25) 1.50, 2.00, 4.00; ρ = 0.00, ± 0.10, ± 0.25, ± 0.50, ± 0.80; α = .05, .20, .50 and .80.

The efficiencies of the two estimators for + ρ are the same as that for $-\rho$ and the biases of the two estimators for + ρ is negative of that for $-\rho$. The tables are quite extensive so only two tablles have been given showing bias and efficiencies of $\hat{\beta}$ and β for N = 5 and N=30. Table 1 shows the bias for negative values of ρ and Table 2 shows the efficiencies.

It is found that for ρ = 0, both the PT estimators are unbiased for β. This is clear from (3.9) and (3.13) because integrand becomes an odd function of u when g = 0. Nature of bias of both the PT estimators is identical. Bias decreases with increasing α and N, and decreasing magnitude of ρ. Bias is quite small except for small sample size and low value of α. So if sample size N≤ 5 , α should be preferably greater than or equal to .50.

If $\frac{\sigma_x^2}{\sigma_y^2} \geq 1.25$ bias of β is usually less that that of $\hat{\beta}$. Thus, in the case of inequally of the variances if σ_x^2 is suspected to be larger, use of β reduces the bias.

The variation of efficiencies of the two estimators is also more or less identical.

Both the estimators are more efficient than the usual estimator if $\sigma_x = \sigma_y$ for all N, α and ρ. For the value of $\frac{\sigma_x^2}{\sigma_y^2}$ = 1.00 and 1.25, the efficiency of $\hat{\beta}$ decreases with increasing α, whereas for a given α, it decreases with decreasing value of $|\rho|$. Efficiency of β in this case also varies almost similarly except that for N =5, it again increases after = $|\rho|$ .5. For other values of $\frac{\sigma_x^2}{\sigma_y^2}$, if for α = .05 efficiency <1, it increases with increasing α and if efficiency >1, it decreases with α. For a given α., in this case the efficiency increases with decreasing value of $|\rho|$. The efficiency of both the estimators is highest if $\sigma_x = \sigma_y$ for all N, and this decreases with increasing N (except for ρ = ± .8).

It is found that $\hat{\beta}$ is more efficient than β if $\frac{\sigma_x^2}{\sigma_y^2} \leq 1$ and $|\rho|$ is large. This range over $|\rho|$ increases with increasing sample sizes. Thus if σ_x^2 is suspected to be larger, use of β will be more beneficial from efficiency point of view also.

If $|\rho| \leq .10$ both the proposed estimators are preferable to the usual estimator if $\frac{\sigma_x^2}{\sigma_y^2} \leq 1.00$ with any value of α.

For small samples (N=5) the efficiency is generally larger than 1 if $|\rho|$ is not very large and $\frac{\sigma_x^2}{\sigma_y^2}$ is not very large.

Increase in N, though increases the highest efficiency but it narrows the range of $|\rho|$ and $\frac{\sigma_x^2}{\sigma_y^2}$ where the estimaators are more efficient than b.

6. *Recommendations.*

On the basis of the above numerical findings we recommend that

(i) Both $\hat{\beta}$ and β may be used with advantage with relatively small N and α i. e. for $N \leq 10$, $\alpha \leq .20$.

(ii) If σ_x^2 is expected to be larger than σ_y^2, β should be preferred to $\hat{\beta}$. $\hat{\beta}$ should be used if σ_x^2 is feared to be less than σ_y^2.

(iii) In case the reduction in bias is considered more important, $\alpha > .50$ should be chosen.

(iv) If $|\rho| < .10$ and σ_x^2 is expected to be smaller than σ_y^2 any value of α may be chosen for gain in efficiency.

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Table 1. Bias of $\hat{\beta}$ and β

N = 5

α	ρ	σ_x^2/σ_y^2 .25	.50	.75	1.00	1.25	1.50	2.00
	–.80	.32 .38	.24 .29	.13 .16	.3(1) .6(1)	–.3(1) –.3(3)	–.8(1) –.4(3)	–.13 –.8(1)
.05	–.50	.24 .29	.16 .20	.18(1) .12	–.3(1) .6(1)	–.6(2) .2(1)	–.3(1) –.4(2)	–.7(1) –.4(1)
	–.10	.5(1) .6(1)	–.3(1) .4(1)	.2(1) .2(1)	–.8(2) .1(1)	.2(3) .7(2)	–.5(2) .1(2)	–.1(1) –.1(2)
	–.80	.7(2) .8(2)	.1(1) .2(1)	.1(1) .1(1)	.3(2) .7(2)	–.4(2) –.6(3)	–.8(2) –.5(2)	–.1(1) –.8(2)
.50	–.50	.8(2) .9(2)	.1(1) .1(1)	.8(2) .1(1)	.3(2) .6(2)	–.9(3) .2(2)	–.4(2) –.9(3)	–.6(2) –.4(2)
	–.10	.2(2) .2(2)	.2(2) .3(2)	.2(2) .2(2)	.7(2) .1(2)	–.1(3) .6(3)	–.6(3) .0(4)	–.1(2) –.6(3)
	–.80	.3(3) .3(3)	.8(3) .9(3)	.7(3) .9(3)	.2(3) –.4(3)	–.2(3) –.9(4)	–.5(3) –.3(3)	–.6(3) –.4(3)
.80	–.50	.4(3) .4(3)	.6(3) .7(3)	.4(3) .6(3)	.2(3) .3(3)	–.1(3) .1(3)	–.2(3) –.1(3)	–.4(3) –.2(3)
	–.10	.1(3) .1(3)	.1(3) .2(3)	.1(3) .1(3)	.0(4) .1(3)	.0(4) .0(4)	.0(4) .0(4)	–.1(3) –.0(4)

N = 30

α	ρ	σ_x^2/σ_y^2 .25	.50	.75	1.00	1.25	1.50	2.00
	–.80	.4(4) .0(4)	.2(1) .3(1)	.6(1) .7(1)	.4(2) .7(2)	–.5(1) –.4(1)	–.5(1) –.5(1)	–.2(1) –.2(1)
.05	–.50	.2(2) .2(2)	.5(1) .5(1)	.5(1) .5(1)	.5(2) .4(2)	–.3(1) –.3(1)	–.5(1) –.4(1)	–.4(1) –.3(1)
	–.10	.1(2) .1(2)	.1(1) .1(1)	.1(1) .1(1)	.1(2) .2(2)	–.6(2) –.5(2)	–.1(1) –.8(2)	–.9(2) –.8(2)
	–.80	.7(7) .8(7)	.4(3) .4(3)	.4(2) .4(2)	.4(3) .7(3)	–.4(2) –.4(2)	–.3(2) –.2(2)	–.3(3) –.3(3)
.50	–.50	.9(5) .1(4)	.2(2) .2(2)	.4(2) .4(2)	.4(3) .8(3)	–.3(2) –.2(2)	–.3(2) –.3(2)	–.1(1) –.1(2)
	–.10	.8(5) .9(5)	.5(3) .5(3)	.8(3) .9(3)	.1(3) .2(3)	–.5(3) –.4(3)	–.7(3) –.6(3)	–.4(3) –.4(3)
	–.80	.2(8) .2(8)	.2(4) .2(4)	.2(3) .2(3)	.2(4) .4(4)	–.2(3) –.2(3)	–.1(3) –.1(3)	–.1(4) –.1(4)
.80	–.50	.3(6) .3(6)	.7(4) .8(4)	.2(3) .2(3)	.3(4) .5(4)	–.2(3) –.1(3)	–.2(3) –.2(3)	–.7(4) –.7(4)
	–.10	.3(6) .3(6)	.2(4) .3(4)	.5(4) .5(4)	.7(5) .1(4)	–.3(4) –.3(4)	–.4(4) –.4(4)	–.2(4) –.2(4)

Note: 1. In each row and column, first value is Bias($\hat{\beta}$) and second one is Bias(β).

2. Integers in bracket are the power of 10 by which the preceding value is to be divided.

Table 2. Efficiencies of $\hat{\beta}$ and β

N = 5

		σ_x^2/σ_y^2							
α	ρ	.25	.50	.75	1.00	1.25	1.50	2.00	4.00
.05	.80	.72 .65	.99 .90	1.35 1.27	1.53 1.50	1.45 1.49	1.25 1.34	.88 1.04	.42 .62
	.50	1.08 1.05	1.29 1.29	1.37 1.41	1.34 1.42	1.25 1.36	1.15 1.28	.96 1.11	.58 .78
	.10	1.26 1.30	1.40 1.48	1.38 1.49	1.30 1.43	1.20 1.34	1.11 1.26	.95 1.12	.63 .83
.50	.80	.99 .98	.98 .97	1.00 1.00	1.03 1.02	1.03 1.03	1.02 1.02	.99 1.00	.95 .97
	.50	1.00 1.00	1.01 1.01	1.01 1.02	1.02 1.02	1.01 1.02	1.01 1.00	1.00 .98	.97 .98
	.10	1.01 1.01	1.01 1.01	1.02 1.02	1.01 1.02	1.01 1.02	1.00 1.01	.99 1.00	.97 .98
.80	.80	1.00- 1.00-	1.00- 1.00-	1.00 1.00-	1.00 1.00	1.00 1.00	1.00- 1.00	1.00- 1.00	1.00- 1.00-
	.50	1.00- 1.00-	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00- 1.00	1.00- 1.00-
	.10	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00- 1.00	1.00- 1.00-

Note : In each column against a value of ρ, the first entry is e_1 and second one is e_2.

'-' indicates actual value below 1 (becomes 1 after rounding to second decimal place).

APPENDIX

Series Expressions for Bias and MSE

Lemma 2.1 :

$$I_1' = C_1' \sum_{r=0}^{\infty} C_r \sum_{j=0}^{r} c_{rj}\, I(rj)\, h^{j+1} \Big[(j+1)^{-1} B(n_{j1}-1,\ q_{rj}-1/2)$$

$$\left\{ F\,{}^{1:2;1}_{1:1;0}\left(\begin{matrix} j1:\ 1/2,\ n_{j1}-1;\ p; \\ j1+1: n_{r1};\ -\ ; \end{matrix}\ h^2, h^2 \right) - F\,{}^{1:2;1}_{1:1;0}\left(\begin{matrix} j1:\ 1,\ n_{j1}-1;\ p; \\ j1+1: n_{r1};\ -\ ; \end{matrix}\ h^2, h^2 \right) \right\}$$

$$-\frac{h}{j+2} B\left(n_{jo}, q_{rj}-\frac{1}{2}\right) F\,{}^{1:2;1}_{1:1;0}\left(\begin{matrix} j2:\ 1,\ n_{jo};\ p; \\ j2+1: n_{r2};\ -\ ; \end{matrix}\ h^2, h^2 \right) \Big]$$

where $s = \dfrac{\sigma_x^2}{\sigma_y^2}$,

$$C_1' = \frac{2^{N-2} (1-\rho^2)^{\frac{N-1}{2}}}{N-2} \left(\frac{\sqrt{S}}{1+S}\right)^{N-1}, \quad C_r = \frac{\Gamma(N+r-1)}{\Gamma(r+1)} (2\rho)^r \left(\frac{\sqrt{S}}{1+S}\right)^r$$

$$C_{rj} = \binom{r}{j} (-2\rho)^{-j} \left(\frac{S-1}{\sqrt{S}}\right)^j, \quad q_{rj} = \frac{r-j+3}{2}, \quad n_{ji} = \frac{N+j+1}{2},$$

$$n_{ri} = \frac{N+r+i}{2}, \quad ji = \frac{j+i}{2}, \quad p = \frac{4-N}{2}, \quad I(rj) = \begin{cases} 1 & \text{if}(r-j) \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

$$F\,{}^{1:2;1}_{1:1;0}\left(\begin{matrix} f;\ p,\ q;\ r; \\ f+1:\ s;\ -\ ; \end{matrix}\ h, h \right)$$

is the Kamp de Feriet function (cf. Rainville [19])

${}_2F_1$ (a, b; c; x) and B(a, b) are the hypergeometric and the beta functions respectively.

The exponential term in (3.9) is expanded in series form before solving inner most integral in v as gamma integral. Next putting $u=\cos\theta$ and using results from Gradshteyn and Ryzhik [6] pp 286 and 389 on the two terms in inner integral on θ, we get integrals on t^2 with range as (O, h^2).

Result on page 193 in Exton [5] is applied now and the stated result in Lemma 2.1 follows.

Any further details may be furnished by the first author on request.

Lemma 2.2 :

$$I_2' = C_1' \sum_{r=0}^{\infty} C_r \sum_{j=0}^{r} C_{rj} (1 - I(rj)\, h)^{j+1} \left[(j+1)^{-1} B(n_{j1} - 1,\ q_{rj}) \right.$$

$$\left\{ F \begin{array}{c} 1:2;1 \\ 1:1;0 \end{array} \left(\begin{array}{l} j1:\ 1,\ n_{j1} - 1;\ p; \\ j1 + 1:\ n_{r2};\ -\ ; \end{array} \ h^2,\ h^2 \right) \right\}$$

$$- \frac{2h}{j+2} B(n_{jo}, q_{rj})\ F \begin{array}{c} 1:2;1 \\ 1:1;0 \end{array} \left(\begin{array}{l} j2:\ 1,\ n_{jo};\ p; \\ j2 + 1:\ n_{r3};\ -\ ; \end{array} \ h^2,\ h^2 \right)$$

$$\left. - \frac{h^2}{j+3} B(n_{j1}, q_{rj})\ F \begin{array}{c} 1:2;1 \\ 1:1;0 \end{array} \left(\begin{array}{l} j3:\ 2,\ n_{j1};\ p; \\ j3 + 1:\ n_{r4};\ -\ ; \end{array} \ h^2,\ h^2 \right) \right]$$

Proof is similar to that of lemma 2.1.

Expressions for J_1' and J_2' also follow similarly.